Aufgabe 5

$u_{tt} = u_{xx}$ mit in $0 \le x \le 1$ und $t \ge 0$

Anfangsbedingungen $u(x,0) = \begin{cases} 2x & \text{für } x \in [0, 0.5] \\ 2-2x & \text{für } x \in [0.5, 1] \end{cases}$ und $u_t(x,0) = 0$

Randbedingungen $u(0,t) = u(1,t) = 0$

Ansatz der Separation der Variablen:

$$u(x,t) = T(t)\, X(x).$$

Einsetzen in die PDGL $u_{tt} = u_{xx}$ liefert

$$T''(t)\, X(x) = T(t) \cdot X''(x)$$

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\omega^2 \quad \text{... negativ wegen Null-Randbedingungen}$$

$T''(t) + \omega^2 T(t) = 0$ hat allg. Lsg. $T(t) = a\sin(\omega t) + b\cos(\omega t)$

$$T'(t) = a\omega\cos(\omega t) - b\omega\sin(\omega t)$$

$$u_t(x,0) = (a\omega \cdot 1 - 0) \cdot X(x) \overset{!}{=} 0 \Rightarrow \underline{a = 0}.$$

$X''(x) + \omega^2 X(x) = 0$ hat allg. Lsg. $X(x) = c\sin(\omega x) + d\cos(\omega x)$

$$u(0,t) = T(t) \cdot (0 + d \cdot 1) \overset{!}{=} 0 \Rightarrow \underline{d = 0}.$$

$$u(1,t) = T(t) \cdot (c \cdot \sin(\omega \cdot 1)) \overset{!}{=} 0 \Rightarrow \sin(\omega) = 0 \Rightarrow$$

$$\omega = n\pi \text{ für } n = 1, 2, 3, \ldots$$

Insgesamt: $\underline{u(x,t) = \sum_{n=1}^{\infty} c_n \cos(n\pi t) \sin(n\pi x).}$

Anpassen an die Anfangsbedingung $u(x,0) = \begin{cases} 2x & \text{für } x \in [0, 0.5] \\ 2-2x & \text{für } x \in [0.5, 1] \end{cases}$:

$$\sum_{n=1}^{\infty} c_n \cdot 1 \cdot \sin(n\pi x) = \begin{cases} \text{\%} \\ \text{\%} \end{cases} \quad \Big| \int_0^1 \ldots \cdot \sin(m\pi x)\, dx$$

$$\frac{1}{2} c_m = \int_0^{0.5} 2x \sin(m\pi x)\, dx + \int_{0.5}^{1} (2-2x) \sin(m\pi x)\, dx$$

Nebenrechnungen: $\int_0^{0.5} \underbrace{x}_{f} \cdot \underbrace{\sin(m\pi x)}_{g'}\, dx = \underbrace{x \cdot \frac{-\cos(m\pi x)}{m\pi}\Big|_0^{0.5}}_{f \cdot g} - \underbrace{\int_0^{0.5} 1 \cdot \frac{-\cos(m\pi x)}{m\pi}\, dx}_{\int f' \cdot g}$

$$= -\frac{1}{2m\pi}\cos\left(m\frac{\pi}{2}\right) - 0 + \frac{1}{(m\pi)^2}\sin(m\pi x)\Big|_0^{0.5} =$$

$$= -\frac{1}{2m\pi}\cos\left(m\frac{\pi}{2}\right) + \frac{1}{(m\pi)^2}\sin\left(m\frac{\pi}{2}\right).$$

$$\int_{0,5}^{1} (1-x)\sin(m\pi x)\,dx = \int_{0,5}^{1} \sin(m\pi x)\,dx - \int_{0,5}^{1} \underbrace{x}_{f}\cdot \underbrace{\sin(m\pi x)}_{g'}\,dx =$$

$$= \frac{-\cos(m\pi x)}{m\pi}\Bigg|_{0,5}^{1} - \left[\underbrace{x}_{f} \cdot \underbrace{\frac{-\cos(m\pi x)}{m\pi}}_{g}\Bigg|_{0,5}^{1} - \int_{0,5}^{1} \underbrace{1}_{f'} \cdot \underbrace{\frac{-\cos(m\pi x)}{m\pi}}_{g}\,dx \right] =$$

$$= -\frac{\cos(m\pi)}{m\pi} + \frac{\cos(m\frac{\pi}{2})}{m\pi} + \frac{\cos(m\pi)}{m\pi} - \frac{1}{2m\pi}\cos(m\tfrac{\pi}{2})$$

$$- \frac{\sin(m\pi x)}{(m\pi)^2}\Bigg|_{0,5}^{1} = \frac{1}{2m\pi}\cos(m\tfrac{\pi}{2}) - \left[0 - \frac{\sin(m\frac{\pi}{2})}{(m\pi)^2}\right]$$

Insgesamt: $\frac{1}{2}c_m = 2\left[0 + 2\frac{\sin(m\frac{\pi}{2})}{(m\pi)^2}\right]$

$$c_m = 8 \cdot \frac{\sin(m\pi/2)}{(m\pi)^2}$$

Die Lösung lautet daher $\underline{\underline{u(x,t) = \sum_{n=1}^{\infty} 8\,\frac{\sin(n\pi/2)}{(n\pi)^2}\cdot\cos(n\pi t)\cdot\sin(n\pi x)}}$.