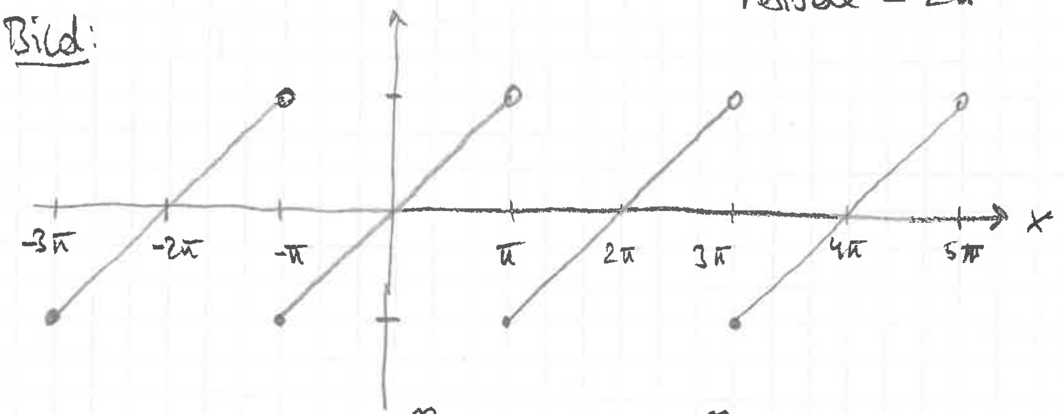
D **Bsp:** <u>Sägezahnfkt.</u>

$f(x) = x$ für $x \in [-\pi, \pi)$ periodisch fortgesetzt,
Periode $= 2\pi$

<u>Bild:</u>



Ziel: $$\underbrace{f(x)}_{\text{ungerade}} = \underbrace{\frac{a_0}{2}}_{\text{ger.}} + \underbrace{\sum_{n=1}^{\infty} a_n \cos(nx)}_{\text{ger.}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin(nx)}_{\text{unger.}}$$

$$\Rightarrow a_0 = 0, \quad a_n = 0 \quad \forall n \geq 1$$

$$b_n = \frac{1}{\pi} \int_{\text{Periode}} f(x) \sin(nx)\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{x}_{f} \cdot \underbrace{\sin(nx)}_{g'}\, dx \underset{\text{part. Int.}}{=}$$

$$= \frac{1}{\pi} \left[\underbrace{\left(x \cdot \frac{-\cos(nx)}{n} \right)}_{f \cdot g} \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \underbrace{\frac{-\cos(nx)}{n}}_{f' \cdot g} dx \right] =$$

$$= \frac{1}{n \cdot \pi} \left[-\pi \cdot \cos(n\pi) - \pi \cos(-n\pi) + \frac{\sin(nx)}{n} \Big|_{-\pi}^{\pi} \right] =$$

$$= \frac{1}{n\pi} \left[-2\pi \cos(n\pi) + 0 \right] = -\frac{2}{n} \cos(n\pi) = -\frac{2}{n} (-1)^n$$

n	b_n
1	$\frac{2}{1} = 2$
2	$-\frac{2}{2} = -1$
3	$\frac{2}{3} = \frac{2}{3}$
4	$-\frac{2}{4} = -\frac{1}{2}$
$\vdots$	$\vdots$

$$\underline{f(x)} = 2\sin(x) - \sin(2x) + \frac{2}{3}\sin(3x) - \frac{1}{2}\sin(4x) + \ldots$$

$\rightarrow$ Code.

Verallgemeinerungen und Umformulierung nach $\mathbb{C}$

▷ $2\pi \to T$: Variable t statt x

$$f(t) = \frac{a_0}{2} + \sum a_n \underbrace{\cos(nt)} + \sum b_n \underbrace{\sin(nt)}$$

Periode $= 2\pi$

$\to$ Periode T: $\cos(n \cdot \underbrace{\tfrac{2\pi}{T}}_{\omega} \cdot t)$ und $\sin(n \cdot \tfrac{2\pi}{T} \cdot t)$

$\boxed{\omega = \frac{2\pi}{T}}$

ω ... Grundkreisfrequenz, Grundschwing.

$n \cdot \omega$ ... Kreisfrequenzen der Oberschwing. für $n = 2, 3, 4, \ldots$

$$f(t) = \frac{a_0}{2} + \sum a_n \cos(n\omega t) + \sum b_n \sin(n\omega t)$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(t)\,dt \;\to\; a_0 = \frac{2}{T} \int_0^T f(t)\,dt$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos(nt)\,dt \to a_n = \frac{2}{T} \int_0^T f(t) \cos(n\omega t)\,dt$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin(nt)\,dt \to b_n = \frac{2}{T} \int_0^T f(t) \sin(n\omega t)\,dt$$

▷ reell $\to$ komplex:

WH: $\boxed{e^{i\varphi} = \cos(\varphi) + i\sin\varphi}$ $\to \cos\varphi = \frac{1}{2}(e^{i\varphi} + e^{-i\varphi})$

gerade ung. Anteil

$\to \sin\varphi = -\frac{i}{2}(e^{i\varphi} - e^{-i\varphi})$

hier: $\left.\begin{array}{l} \cos(n\omega t) = \frac{1}{2}(e^{in\omega t} + e^{-in\omega t}) \\ \sin(n\omega t) = -\frac{i}{2}(e^{in\omega t} - e^{-in\omega t}) \end{array}\right\}$ einsetzen

$$f(t) = \frac{a_0}{2} + \sum a_n \frac{1}{2}(e^{+} + e^{-}) + \sum b_n \left(-\frac{i}{2}\right)(e^{+} - e^{-})$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{1}{2}(a_n - ib_n) e^{in\omega t} + \sum_{n=1}^{\infty} \frac{1}{2}(a_n + ib_n) e^{-in\omega t}$$

Def: $c_0 = \frac{a_0}{2}$, $c_n = \frac{1}{2}(a_n - ib_n)$, $c_{-n} = \frac{1}{2}(a_n + ib_n)$

$$f(t) = c_0 + \sum_{n \geq 1} c_n e^{in\omega t} + \sum_{n \geq 1} c_{-n} e^{-in\omega t} =$$

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}}.$$

Analoge Rechnung führt zu:

$$c_n = \frac{1}{2}(a_n - i b_n) = \ldots = \frac{1}{T}\int_0^T f(t)\, e^{-in\omega t}\, dt. \quad \text{etc}$$

<u>Zsfg:</u> <u>komplexe Fourierreihe:</u>

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{in\omega t} \quad \text{mit } c_n = \frac{1}{T}\int_0^T f(t)\, e^{-in\omega t}\, dt}$$

▷ <u>$T \to \infty$:</u>

$$f(t) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{T}\int_{-\frac{T}{2}}^{T/2} f(t)\, e^{-in\omega t}\, dt\right) e^{in\omega t}$$

$\omega_n := n\omega$, $\Delta\omega_n = \omega = \frac{2\pi}{T}$

$$= \frac{1}{2\pi}\sum \underbrace{\left(\int_{-T/2}^{T/2} f(t)\, e^{-i\omega_n t}\, dt\right)}_{=: F(\omega_n)} e^{i\omega_n t}\, \Delta\omega_n$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} F(\omega_n)\, e^{i\omega_n t}\, \Delta\omega_n$$

$\downarrow \lim_{T\to\infty}$: $\omega_n \to \omega \in \mathbb{R}$, $\Delta\omega_n \to d\omega$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{+i\omega t}\, d\omega$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

vgl.: Laplace $\quad F(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt$

$i\omega \mathrel{\hat=} s$

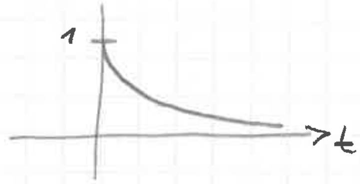
Grenzen anders!

bzw falls $f(t) = 0$ im Negativen, dann gleiche Grenzen.

$$\boxed{\begin{array}{c} f(t) \xrightarrow{\mathcal{F}} \mathcal{F}\{f\}(\omega) = F(\omega) \\ \text{Fouriertrafo} \\ f(t) = \mathcal{F}^{-1}(F)(t) \xleftarrow{\mathcal{F}^{-1}} F(\omega) \\ \text{inverse Fouriertrafo} \end{array}}$$

$$\mathcal{F}^{-1}(F)(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega$$

Bsp: $f(t) = \begin{cases} e^{-at} & \text{für } t \geq 0 \\ 0 & \text{für } t < 0 \end{cases}$



$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt = \int_0^{\infty} e^{-at} e^{-i\omega t} dt = \int_0^{\infty} e^{-(a+i\omega)t} dt =$$

$$= \frac{1}{-(a+i\omega)} e^{-(a+i\omega)t} \Big|_0^{\infty} = 0 + \frac{1}{a+i\omega} = \underline{\underline{\frac{1}{a+i\omega}}}$$

vgl: $\mathcal{L}(e^{-at}) = \frac{1}{a+s} \Rightarrow \underline{s \,\hat{=}\, i\omega}$.

- **Eigenschaften der $\mathcal{F}$ und $\mathcal{F}^{-1}$: Auswahl**

Linear: $\underline{\mathcal{F}(c_1 f_1 + c_2 f_2) = c_1 \mathcal{F}(f_1) + c_2 \mathcal{F}(f_2)}$

genauso $\mathcal{F}^{-1}$

Zeitverschiebung: $\mathcal{F}\{f(t-t_0)\} = \underbrace{e^{-i\omega t_0}}_{\text{Phase}} F(\omega)$

Frequenzverschiebung:

$$\mathcal{F}\{e^{i\omega_0 t} f(t)\} = F(\omega - \omega_0)$$

Modulation $\mathcal{F}\{\cos(\omega_0 t) f(t)\} = \frac{1}{2}\left[F(\omega-\omega_0) + F(\omega+\omega_0)\right]$

etc.

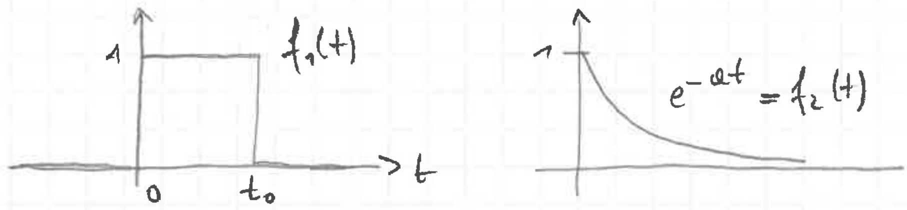
Ableitung: $\underline{\mathcal{F}\{f'(t)\} = i\omega F(\omega)}$.

Faltung: $\mathcal{F}\{(f_1 * f_2)(t)\} = F_1(\omega) \cdot F_2(\omega)$

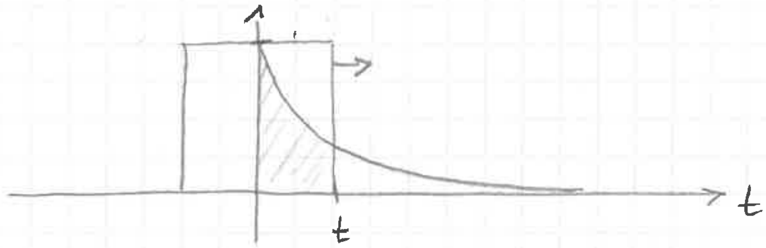
mit $(f_1 * f_2)(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$ ... Faltung von f_1 und f_2

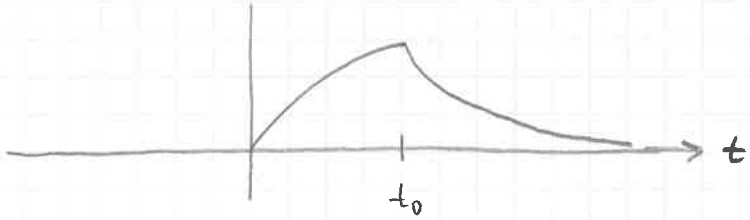
$$= (f_2 * f_1)(t) = \int f_2(\tau) f_1(t-\tau) d\tau$$

Bsp:

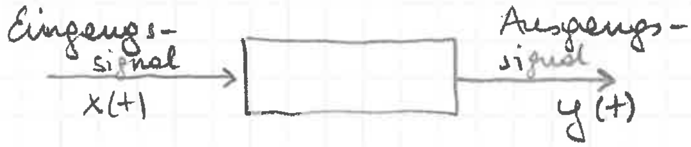


$(f_1 * f_2)(t)$

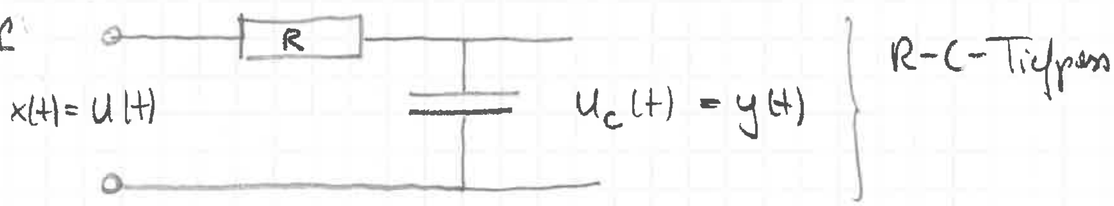




Anwendungen: Signalanalyse, Signalübertragungssysteme, ...



Bsp:



$$U_R(t) + U_c(t) = U(t)$$

$$R \cdot I(t) + U_c(t) = U(t)$$

$$R \cdot C\, U_c'(t) + U_c(t) = U(t)$$

$\downarrow$ allg.

$$a \cdot y'(t) + y(t) = x(t) \qquad | \ \mathcal{F}$$

$$a i\omega Y(\omega) + Y(\omega) = X(\omega)$$

$$(a i\omega + 1)\, Y(\omega) = X(\omega)$$

Frequenzgang: $H(\omega) := \dfrac{Y(\omega)}{X(\omega)} = \dfrac{1}{a i\omega + 1} \overset{RC = a}{=}$

im Bsp: $= \underline{\dfrac{1}{RCi\omega + 1}}$

Allg: $Y(\omega) = H(\omega)\, X(\omega) \qquad | \mathcal{F}^{-1}$

$$y(t) = (h * x)(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

▷ Vorgehensweise: $H(\omega)$ beschreibt das System $\rightarrow \boxed{H(\omega)} \rightarrow$

$H(\omega)$ bekannt $\longrightarrow$ $h(t)$ berechnen $\longrightarrow$ $y(t) = (h*x)(t)$ berechnen

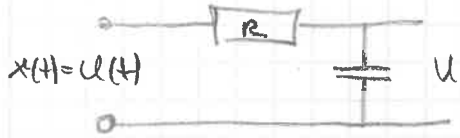
▷ Bsp: $x(t) = x_0 e^{i\omega t}$ ... komplexe harmonische Schwingung

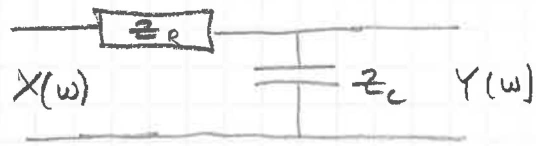
$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x_0 e^{i\omega(t-\tau)}\, d\tau =$$

$$= x_0 e^{i\omega t} \cdot \int_{-\infty}^{\infty} h(\tau)\, e^{-i\omega\tau}\, d\tau = x_0 e^{i\omega t} H(\omega)$$

$$y(t) = H(\omega) \cdot x(t)$$

▷ Statt $H(\omega)$ aus DGL aus Impedanzen Z





Spannungsteiler: $H(\omega) = \dfrac{Y(\omega)}{X(\omega)} = \dfrac{Z_c}{Z_R + Z_c} = \dfrac{1/i\omega C}{R + 1/i\omega C} =$

$$H(\omega) = \frac{1}{i\omega RC + 1}$$ = dasselbe wie früher.